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LETTER TO THE EDITOR

Tunnelling process for non-Hermitian systems: the complex-frequency inverted oscillator

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Abstract. We study the tunnel effect for an inverted oscillator with complex frequency. The solution of the corresponding non-Hermitian (NH) Schrödinger equation is found by the evolution operator method, based on the SU(1, 1) structure of the Hamiltonian and the Wei–Norman theorem. We put forward a generalization of dwell time for NH systems built up from their biorthonormal states. The resulting tunnelling time turns out to be complex.

A problem in quantum mechanics which is still controversial, in spite of its apparent simplicity, *is the seemingly innocuous question: how long does it take a particle to* tunnel through a potential barrier? Renewed interest in such a question has been recently revived [1, 2], in connection with the experimental applications (e.g. to high-energy physics) of tunnelling processes in semiconductors.

Since most of the physical systems one encounters are not isolated but somehow interact with their surroundings, in previous work [3, 4] we have investigated the problem of tunnelling for open systems described by time-dependent Hamiltonians of the Caldirola–Kanai [5] (CK) type. However, CK oscillators cannot be considered as genuine dissipative systems [6]. As is well known, systems exhibiting a truly dissipative behaviour (possessing decaying states) are phenomenologically described by non-Hermitian (NH) Hamiltonians [7]. In the last few years, rigorous methods have been developed to study the time evolution of non-Hermitian systems [8, 9].

In this letter, we discuss the tunnelling effect for a prototype NH Hamiltonian, i.e. the inverted oscillator with complex frequency

$$H=\frac{p^2}{2m}-\frac{m}{2}\,\omega^2\,\mathrm{e}^{2i\theta}q^2. \tag{1}$$

As is well known, the main advantage of the inverted oscillator is that for such a potential the tunnelling time does not diverge at the threshold in semiclassical limit analysis [10], contrary to the case of square-barrier potentials [1, 2]. The prototype NH

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Hamiltonian (1) is just a very simple, special case of the generalized parametric oscillator with complex coefficients that is often encountered as a phenomenological model in dissipative optical processes (e.g. electromagnetic pulse propagation in a free electron laser [11]).

In order to solve the time-dependent non-Hermitian Schrödinger (NHS) equation for Hamiltonian (1), we assume that the state of our system is initially represented by the wavepacket

$$\psi_{(q_0,p_0)}(q,0)=(2\pi\sigma^2)^{-1/4}\exp\left[-\frac{1}{4\sigma^2}(q-q_0)^2+\frac{\mathrm{i}}{\hbar}p_0 q\right]. \tag{2}$$

The solution of the NHS equation is then obtained by propagating the above initial state according to the relation

$$\psi_{(q_0,p_0)}(q,t)=\mathcal{U}(q,t)\psi_{(q_0,p_0)}(q,0). \tag{3}$$

Exploiting the Wei–Norman (WN) theorem [12] and the SU(1, 1) structure of Hamiltonian (1), we can write the evolution operator in the ordered form [13]:

$$\mathcal{U}(q,t)=\mathrm{e}^{\Lambda(t)}\,\mathrm{e}^{a(t)q^2}\,\mathrm{e}^{b(t)q(\partial/\partial q)}\,\mathrm{e}^{c(t)(\partial^2/\partial q^2)} \tag{4}$$

where the WN characteristic functions are given by

$$\Lambda(t)=\frac{b(t)}{2}=-\tfrac{1}{2}\ln\cosh(\omega t\mathrm{e}^{\mathrm{i}\theta}) \tag{5}$$

$$a(t)=\frac{\mathrm{i}m\omega}{2\hbar}\,\mathrm{e}^{\mathrm{i}\theta}\tanh(\omega t\,\mathrm{e}^{\mathrm{i}\theta}); \tag{6}$$

$$c(t)=\frac{\mathrm{i}\hbar}{2m\omega}\,\mathrm{e}^{-\mathrm{i}\theta}\tanh(\omega t\,\mathrm{e}^{\mathrm{i}\theta}). \tag{7}$$

Therefore the solution (3) reads explicity

$$\psi_{(q_0,p_0)}(q,t)=\frac{1}{(2\pi\sigma^2)^{1/4}}\frac{\mathrm{e}^{\Lambda(t)}}{\sqrt{1+\frac{c(t)}{\sigma^2}}}\exp\left\{a(t)q^2+\mathrm{i}k_0q_0-k^2\sigma^2\right.$$

$$\left.-\frac{1}{4\sigma^2}\frac{[\mathrm{e}^{b(t)}q-q_0+2\mathrm{i}k_0\sigma^2]^2}{1+\frac{c(t)}{\sigma^2}}\right\}. \tag{8}$$

As is well known from the theory of *NH* systems [8, 9] one has to consider also the time-dependent equation for the adjoint Hamiltonian H^+ $(\neq H)$

$$H^+ = \frac{p^2}{2m} - \frac{m}{2}\omega^2 e^{-2i\theta} q^2 \tag{9}$$

whose solution can be written as [9]

$$\chi_{(q_0,p_0)}(q,t) = \bar{\mathcal{U}}(q,t)\chi_{(q_0,p_0)}(q,0). \tag{10}$$

Here, $\bar{\mathcal{U}}(q,t)$ is the evolution operator corresponding to H^+, which reads like (4), with the WN characteristic functions $\bar{a}(t)$, $\bar{b}(t)$, $\bar{c}(t)$ formally obtainable from expressions (5)–(7) by the substitution $\theta \to -\theta$. Therefore solution (10) becomes

$$\chi_{(q_0,p_0)}(q,t) = \frac{1}{(2\pi\sigma^2)^{1/4}} \frac{e^{\Lambda(t,-\theta)}}{\left[1+\frac{c(t,-\theta)}{\sigma^2}\right]^{1/2}} \exp\left\{ a(t,-\theta)q^2 + ik_0q_0 - k^2\sigma^2 \right.$$
$$\left. - \frac{1}{4\sigma^2} \frac{[e^{b(t,-\theta)}q - q_0 + 2ik_0\sigma^2]^2}{1+\frac{c(t,-\theta)}{\sigma^2}} \right\}. \tag{11}$$

It is easy to show that the operators $\mathcal{U}$, $\bar{\mathcal{U}}$ are bi-unitary [9], namely

$$\mathcal{U}\bar{\mathcal{U}}^+ = \bar{\mathcal{U}}^+\mathcal{U} = I \tag{12}$$

and

$$\langle \chi_{(q_0,p_0)}(q,t) | \psi_{(q_0,p_0)}(q,t) \rangle = 1 \tag{13}$$

i.e. the states χ, ψ are bi-orthonormal at any time.

As is well known from the theory of tunnelling processes [1, 2], the dwell (or sojourn) time for Hermitian systems is defined as

$$\tau = \int_{-\infty}^{+\infty} dt \int_{\alpha}^{\beta} |\psi(q,t)|^2\, dq \tag{14}$$

where (α, β) is an interval containing the barrier. Then, it is quite natural to generalize the above definition for open systems described by a NH Hamiltonian as follows:

$$\tau_{NH} = \int_{-\infty}^{+\infty} dt \int_{\alpha}^{\beta} \chi^*(q,t)\psi(q,t)\, dq$$
$$= \int_{-\infty}^{+\infty} dt \int_{\alpha}^{\beta} \rho_{NH}(q,t)\, dq. \tag{15}$$

Therefore, for the complex-frequency inverted oscillator (1) we obtain, by (8) and (11):

$$\rho_{NH}(q,t)=\chi^*(q,t)\psi(q,t)$$
$$=\frac{1}{(2\pi\sigma^2)^{1/2}}\frac{e^{b(t)}}{\sqrt{1-\frac{c^2}{\sigma^4}}}\exp\left\{-\frac{1}{2\sigma^2}\frac{[q\,e^{b(t)}-q_0+2ik_0c(t)]^2}{1-\frac{c^2}{\sigma^4}}\right\}$$
$$=\frac{1}{(2\pi\sigma^2)^{1/2}}\frac{e^{b(t)}}{\sqrt{1-\frac{c^2}{\sigma^4}}}\exp\left\{-\frac{e^{2b(t)}}{2\sigma^2}\frac{(q-q_{max})^2}{1-\frac{c^2(t)}{\sigma^4}}\right\} \tag{16}$$

with

$$q_{max}=e^{-b(t)}\left[q_0+\frac{p_0}{m\omega}e^{-i\theta}\tanh(\omega t\,e^{i\theta})\right]. \tag{17}$$

Finally, setting

$$\Gamma^2(t)=e^{-2b(t)}\left[1+\frac{\lambda^4}{c^4}e^{-i\theta}\tanh^2(\omega t\,e^{i\theta})\right] \tag{18}$$

where $\lambda=(\hbar/2m\omega)^{1/2}$ is a characteristic length of the system, we obtain

$$\rho_{NH}(q,t)=[2\pi\sigma^2\Gamma^2(t)]^{-1/2}\exp\left[-\frac{(q-q_{max})^2}{2\sigma^2\Gamma^2(t)}\right]. \tag{19}$$

Then, the NH sojourn time (15) reads

$$\tau_{NH}=\frac{1}{2}\int_{-\infty}^{+\infty}\left\{\mathrm{Erf}\left[\frac{\beta-q_{max}}{\sqrt{2}\sigma\Gamma(t)}\right]-\mathrm{Erf}\left[\frac{\alpha-q_{max}}{\sqrt{2}\sigma\Gamma(t)}\right]\right\}dt \tag{20}$$

with Erf(·) being the error function.

In the case of an extended wavepacket, i.e. $\sigma\gg\lambda$ and

$$\frac{\beta-q_{max}}{\sqrt{2}\,e^{-b(t)}}\ll\sigma \tag{21a}$$

$$\frac{\alpha-q_{max}}{\sqrt{2}\,e^{-b(t)}}\ll\sigma \tag{21b}$$

we obtain, to a first approximation ($L=\beta-\alpha$):

$$\tau_{NH}\simeq\frac{L}{2\sqrt{2}\sigma}\int_{-\infty}^{+\infty}e^{b(t)}\,dt$$
$$=\frac{L}{2\sqrt{2}\sigma}\int_{-\infty}^{+\infty}\frac{dt}{\cosh(\omega t\,e^{i\theta})} \tag{22}$$
$$=\frac{L}{2\sqrt{2}\sigma\omega}\int_{-\infty}^{+\infty}\frac{du}{\cosh(ue^{i\theta})}$$

($u=\omega t$).

The integral (22) diverges for $\theta=\pi/2$, because $\cosh(u e^{i\pi/2})$ reduces to $\cos u$ and has poles at $u=\pi(n+\frac{1}{2})$. This fact is connected with the remarks given by Barton [10] concerning the existence of singularities—which occur in the transmission and reflection amplitudes—in the case of the ordinary inverted oscillator. Indeed the Hamiltonian (1) for $\theta=\pi/2$ reduces to that of the usual harmonic oscillator.

In order to avoid the poles, we must take $0\leqslant\theta<\pi/2$. In this case, equation (22) becomes simply

$$\tau_{\rm NH}=\frac{\pi}{2}\frac{L\,e^{-i\theta}}{\sqrt{2}\sigma\omega}. \tag{23}$$

It is easily seen that for $\theta=0$ one recovers the sojourn time of the usual inverted oscillator [3, 10].

We stress that the sojourn time for the complex frequency inverted oscillator turns out to be complex. This is clearly expected to be a general feature of tunnelling times for NH systems, and is connected with the complex eigenvalues of the energy, needed to ensure the occurrence of truly decaying states.

The possibility of complex tunnelling times for Hermitian systems has been widely discussed in the literature [14–17]. However, in the Hermitian case the very origin and meaning of complex tunnelling times remains largely unexplained (17). On the contrary, for the NH tunnel effect, the occurrence of complex times is quite easily understood on the basis of energy non-conservation and the time–energy uncertainty principle.

In particular, their imaginary part accounts for the influence of additional degrees of freedom [18], which (unlike the case of Hermitian systems) are always present (although unspecified) in the phenomenological representation of dissipative systems through non-Hermitian Hamiltonians. In this connection, a possible interpretation of the complex tunnelling time for open systems is that the real part corresponds to an 'intrinsic' traversal time related to the potential barrier, whereas the imaginary part represents a characteristic time of interaction between the system and its surrounding. Of course, this does not mean at all that the real part of the complex time must coincide with the tunnelling time of the system in absence of dissipation, because it is expected on physical grounds that the coupling to external degrees of freedom does affect also, in general, the time spent by the system in crossing the barrier. In our opinion a deeper insight to this problem may come from the introduction of non-Hermitian sojourn time operators, along the lines of [17]. Such a point will be discussed elsewhere.

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References

[1] Hauge E H and Stovneng A J 1989 *Rev. Mod. Phys.* **61** 917 and references therein
[2] Olkhovsky V S and Recami E 1992 *Phys. Rep.: Math. Gen.* **214** 339 and references therein
[3] Baskoutas S and Jannussis A: 1992 *J. Phys. A: Math Gen.* **25** L1299
[4] Baskoutas S, Jannussis A and Mignani R 1993 *J. Phys. A: Math. Gen.* in press
[5] Caldirola P 1941 *Nuovo Cimento* **18** 393
Kanai E 1948 *Prog. Theor. Phys.* **3** 440

[6] Crespo G, Proto A N, Plastino A and Otero D 1990 *Phys. Rev.* A **42** 3608
[7] Fonda L, Ghirardi G C and Rimini A 1978 *Rep. Prog. Phys.* **41** 587
[8] Baker H C 1983 *Phys. Rev. Lett.* **50** 1579; 1984 *Phys. Rev.* A **30** 773
Baker H C and Singleton R L, Jr 1990 *Phys. Rev.* A **42** 10
[9] Dattoli G, Torre A and Mignani R 1990 *Phys. Rev.* A **42** 1467
[10] Barton G 1986 *Ann. Phys.* **166** 322
[11] Dattoli G, Torre A, Reali G and Richetta M 1988 *Nuovo Cimento* B **101** 585 and references therein
[12] Wei J and Norman E 1963 *J. Math. Phys.* **4** 575
Dattoli G, Gallardo J C and Torre A 1988 *Riv. Nuovo Cimento* **11** 1
[13] Baskoutas S, Jannussis A and Mignani R 1992 *Phys. Lett.* **164A** 17
[14] Sokolovski D and Baskin L M 1987 *Phys. Rev.* A **36** 4604
[15] Leavens C R and Aers G C 1987 *Solid State Commun* **63** 110
[16] Recami E, Rodrigues W A, Jr and Smrz P 1983 *Hadronic J.* **6** 1773 and references therein
[17] Jaworski W and Wardlaw D M 1991 *Phys. Rev.* A **43** 5137 and references therein
[18] Pollack E and Miller W H 1984 *Phys. Rev. Lett.* **53** 115
Pollak E 1985 *J. Chem. Phys.* **83** 1111